LINEAR PROGRAMMING

1. Introduction.

A linear programming problem may be defined as the problem of *maximizing or minimizing a linear function subject to linear constraints.* The constraints may be equalities or inequalities. Here is a simple example.

Find numbers x_1 and x_2 that maximize the sum $x_1 + x_2$ subject to the constraints $x_1 \geq 0$, $x_2 \geq 0$, and

$$\begin{aligned} x_1 + 2x_2 &\leq 4 \\ 4x_1 + 2x_2 &\leq 12 \\ -x_1 + x_2 &\leq 1 \end{aligned}$$

In this problem there are two unknowns, and five constraints. All the constraints are inequalities and they are all linear in the sense that each involves an inequality in some linear function of the variables. The first two constraints, $x_1 \geq 0$ and $x_2 \geq 0$, are special. These are called *nonnegativity constraints* and are often found in linear programming problems. The other constraints are then called the *main constraints.* The function to be maximized (or minimized) is called the *objective function.* Here, the objective function is $x_1 + x_2$.

Since there are only two variables, we can solve this problem by graphing the set of points in the plane that satisfies all the constraints (called the constraint set) and then finding which point of this set maximizes the value of the objective function. Each inequality constraint is satisfied by a half-plane of points, and the constraint set is the intersection of all the half-planes. In the present example, the constraint set is the five-sided figure shaded in Figure 1.

We seek the point (x_1, x_2), that achieves the maximum of $x_1 + x_2$ as (x_1, x_2) ranges over this constraint set. The function $x_1 + x_2$ is constant on lines with slope -1, for example the line $x_1 + x_2 = 1$, and as we move this line further from the origin up and to the right, the value of $x_1 + x_2$ increases. Therefore, we seek the line of slope -1 that is farthest from the origin and still touches the constraint set. This occurs at the intersection of the lines $x_1 + 2x_2 = 4$ and $4x_1 + 2x_2 = 12$, namely, $(x_1, x_2) = (8/3, 2/3)$. The value of the objective function there is $(8/3) + (2/3) = 10/3$.

Exercises 1 and 2 can be solved as above by graphing the feasible set.

It is easy to see in general that the objective function, being linear, always takes on its maximum (or minimum) value at a corner point of the constraint set, provided the

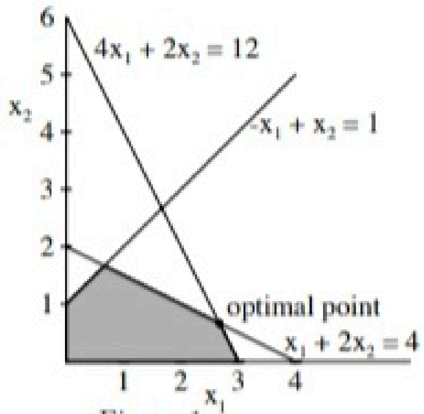

Figure 1.

constraint set is bounded. Occasionally, the maximum occurs along an entire edge or face of the constraint set, but then the maximum occurs at a corner point as well.

Not all linear programming problems are so easily solved. There may be many variables and many constraints. Some variables may be constrained to be nonnegative and others unconstrained. Some of the main constraints may be equalities and others inequalities. However, two classes of problems, called here the *standard maximum problem* and the *standard minimum problem*, play a special role. In these problems, all variables are constrained to be nonnegative, and all main constraints are inequalities.

We are given an m-vector, $\boldsymbol{b} = (b_1, \ldots, b_m)^T$, an n-vector, $\boldsymbol{c} = (c_1, \ldots, c_n)^T$, and an $m \times n$ matrix,

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

of real numbers.

The Standard Maximum Problem: Find an n-vector, $\boldsymbol{x} = (x_1, \ldots, x_n)^T$, to maximize

$$\boldsymbol{c}^T\boldsymbol{x} = c_1x_1 + \cdots + c_nx_n$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned} \qquad (\text{or } \boldsymbol{Ax} \le \boldsymbol{b})$$

and

$$x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0 \qquad (\text{or } \boldsymbol{x} \ge \boldsymbol{0}).$$

The Standard Minimum Problem: Find an m-vector, $\boldsymbol{y} = (y_1, \ldots, y_m)$, to minimize

$$\boldsymbol{y}^T\boldsymbol{b} = y_1b_1 + \cdots + y_mb_m$$

subject to the constraints

$$\begin{aligned} y_1a_{11} + y_2a_{21} + \cdots + y_ma_{m1} &\geq c_1 \\ y_1a_{12} + y_2a_{22} + \cdots + y_ma_{m2} &\geq c_2 \\ &\vdots \\ y_1a_{1n} + y_2a_{2n} + \cdots + y_ma_{mn} &\geq c_n \end{aligned} \qquad (\text{or } \boldsymbol{y}^T\boldsymbol{A} \geq \boldsymbol{c}^T)$$

and

$$y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0 \qquad (\text{or } \boldsymbol{y} \geq \boldsymbol{0}).$$

Note that the main constraints are written as $\leq$ for the standard maximum problem and $\geq$ for the standard minimum problem. The introductory example is a standard maximum problem.

We now present examples of four general linear programming problems. Each of these problems has been extensively studied.

Example 1. The Diet Problem. There are m different types of food, $F_1, \ldots, F_m$, that supply varying quantities of the n nutrients, $N_1, \ldots, N_n$, that are essential to good health. Let c_j be the minimum daily requirement of nutrient, N_j. Let b_i be the price per unit of food, F_i. Let a_{ij} be the amount of nutrient N_j contained in one unit of food F_i. The problem is to supply the required nutrients at minimum cost.

Let y_i be the number of units of food F_i to be purchased per day. The cost per day of such a diet is

$$b_1y_1 + b_2y_2 + \cdots + b_my_m. \tag{1}$$

The amount of nutrient N_j contained in this diet is

$$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m$$

for $j = 1, \ldots, n$. We do not consider such a diet unless all the minimum daily requirements are met, that is, unless

$$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \geq c_j \qquad \text{for } j = 1, \ldots, n. \tag{2}$$

Of course, we cannot purchase a negative amount of food, so we automatically have the constraints

$$y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0. \tag{3}$$

Our problem is: minimize (1) subject to (2) and (3). This is exactly the standard minimum problem.

Example 2. The Transportation Problem. There are I ports, or production plants, $P_1, \ldots, P_I$, that supply a certain commodity, and there are J markets, $M_1, \ldots, M_J$, to which this commodity must be shipped. Port P_i possesses an amount s_i of the commodity ($i = 1, 2, \ldots, I$), and market M_j must receive the amount r_j of the commodity ($j = 1, \ldots, J$). Let b_{ij} be the cost of transporting one unit of the commodity from port P_i to market M_j. The problem is to meet the market requirements at minimum transportation cost.

Let y_{ij} be the quantity of the commodity shipped from port P_i to market M_j. The total transportation cost is

$$\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij}b_{ij}. \tag{4}$$

The amount sent from port P_i is $\sum_{j=1}^{J} y_{ij}$ and since the amount available at port P_i is s_i, we must have

$$\sum_{j=1}^{J} y_{ij} \le s_i \qquad \text{for } i = 1, \ldots, I. \tag{5}$$

The amount sent to market M_j is $\sum_{i=1}^{I} y_{ij}$, and since the amount required there is r_j, we must have

$$\sum_{i=1}^{I} y_{ij} \ge r_j \qquad \text{for } j = 1, \ldots, J. \tag{6}$$

It is assumed that we cannot send a negative amount from P_I to M_j, we have

$$y_{ij} \ge 0 \qquad \text{for } i = 1, \ldots, I \text{ and } j = 1, \ldots, J. \tag{7}$$

Our problem is: minimize (4) subject to (5), (6) and (7).

Let us put this problem in the form of a standard minimum problem. The number of y variables is IJ, so $m = IJ$. But what is n? It is the total number of main constraints. There are $n = I + J$ of them, but some of the constraints are $\ge$ constraints, and some of them are $\le$ constraints. In the standard minimum problem, all constraints are $\ge$. This can be obtained by multiplying the constraints (5) by -1:

$$\sum_{j=1}^{J}(-1)y_{ij} \ge -s_i \qquad \text{for } i = 1, \ldots, I. \tag{5'}$$

The problem "minimize (4) subject to (5′), (6) and (7)" is now in standard form. In Exercise 3, you are asked to write out the matrix $\boldsymbol{A}$ for this problem.

Example 3. The Activity Analysis Problem. There are n activities, $A_1, \ldots, A_n$, that a company may employ, using the available supply of m resources, $R_1, \ldots, R_m$ (labor hours, steel, etc.). Let b_i be the available supply of resource R_i. Let a_{ij} be the amount